REPLY TO
"COMMENT ON 'GRAVITY AND THE POINCARÉ GROUP' "

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Abstract

In the first order form, the model considered by Strobl presents, besides local Lorentz and diffeomorphism invariances, an additional local non-linear symmetry. When the model is realized as a Poincaré gauge theory according to the procedure outlined in Refs.[1,2], the generators of the non-linear symmetry are responsible for the "nasty constraint algebra". We show that not only the Poincaré gauge theoretic formulation of the model is not the cause of the emerging of the undesirable constraint algebra, but actually allows to overcome the problem. In fact one can fix the additional symmetry without breaking the Poincaré gauge symmetry and the diffeomorphisms, so that, after a preliminary Dirac procedure, the remaining constraints uniquely satisfy the Poincaré algebra. After the additional symmetry is fixed, the equations of motion are unaltered. The objections to our method raised by Strobl in Ref.[3] are then immaterial. Some minor points put forward in Ref.[3] are also discussed.

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In Refs.[1,2] we showed that gravity in any dimension with all its couplings to matter can be formulated as a gauge theory of the Poincaré group. In order to reach this goal, a set of auxiliary fields - the Poincaré coordinates $q^a(x)$ - are introduced, and the Poincaré covariant derivative of such fields in the defining representation is identified with the *zweibein* ${V^a}_\mu$,[†]

$$\mathcal{D}_\mu q^a(x) = \partial_\mu q^a + \omega_\mu {\varepsilon^a}_b q^b + {e^a}_\mu \equiv {V^a}_\mu \quad , \tag{1}$$

so that the metric turns out Poincaré gauge invariant.

The spin connection ω_μ and ${e^a}_\mu$ can be combined together as components of the $ISO(1,1)$ gauge potential, $A_\mu = J\omega_\mu + P_a {e^a}_\mu$, where J and P_a are the Lorentz and the translation generators, respectively.

Recently Strobl wrote a Comment [3] criticizing our papers [1,2]. Besides some minor points that we shall briefly discuss below, the main criticism made in Ref.[3] is that if our method to reformulate gravitational theories as Poincaré gauge theories is applied to a given 2 dimensional model of non-Einsteinian gravity, one finds a "nasty" constraint algebra (with structure functions instead of constants) that is not a representation of the gauge group.

In this reply we shall show that the above conclusion is mistaken and that one can suitably apply a canonical Dirac procedure so that the constraints one ends up with precisely satisfy the $ISO(1,1)$ algebra. In fact, in the first order formulation of the model Strobl considers, one has a non-linear local symmetry (specific of the model under consideration), whose generators are responsible for the "nasty" constraint algebra. As we shall see, the Poincaré gauge theoretical formulation not only is not the cause of the emerging of the non-linear algebra, but actually permits to overcome this problem: one can introduce an auxiliary condition that removes the non-linear symmetry without altering the Poincaré gauge symmetry, the diffeomorphism invariance and the equations of motion. Therefore, our formalism can indeed simplify the canonical structure of the model yet preserving all its relevant symmetries.

Before explaining quantitatively these statements, we would like to point out a difference between the standard and our approach to gravity as a gauge theory, difference that was not recognized in Ref. [3]. In Refs. [1, 2], in order to cast gravity as close as possible to any ordinary non-Abelian gauge theory, we did not parametrize the translational part of the Poincaré group in such a way to reproduce general coordinate transformations. Consequently, a gauge transformation does not entail a coordinate transformation. Nevertheless, Poincaré gauge invariant actions turn out to be invariant *also* under diffeomorphism transformations.

The model we shall deal with is given by the following second-order Lagrangian [4]

$$S_{\mathrm{S}} = \int d^2x \, \mathcal{L}_{\mathrm{S}} = \int d^2x \frac{\sqrt{-g}}{4} \left(\gamma R^2 + \beta {T^a}_{\mu\nu} {T_a}^{\mu\nu} + 4\Lambda \right) \quad , \tag{2}$$

where R is the scalar curvature, $\sqrt{-g}R = \epsilon^{\mu\nu} R_{\mu\nu} = \epsilon^{\mu\nu}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)$ and γ, β and Λ are constants. At this stage, ${e^a}_\mu$ is the *zweibein* so that ${T^a}_{\mu\nu} = \partial_\mu {e^a}_\nu - \partial_\nu {e^a}_\mu + {\epsilon^a}_b(\omega_\mu {e^b}_\nu - \omega_\nu {e^b}_\mu)$ is the spacetime torsion. S_{S} is invariant under diffeomorphisms and under local Lorentz transformations.

[†] For later convenience, we shall consider here the 2 dimensional case. For the 3 and 4 dimensional case see Ref. [1]. The conventions on the indices and on the antisymmetric symbols are those of Ref. [2].

By introducing the Lagrange multipliers π_a and π_2, the action (2) can be rewritten in a first order form

$$S_{\rm H} = \int d^2x\, \mathcal{L}_{\rm H} = \int \frac{d^2x}{2}\, \epsilon^{\mu\nu} \left(\pi_2 R_{\mu\nu} + \pi_a T^a{}_{\mu\nu} + \varepsilon_{ab} e^a{}_\mu e^b_\nu E \right) \quad , \tag{3.a}$$

$$E = \frac{1}{4\gamma}(\pi_2)^2 - \frac{1}{2\beta}\pi_a\pi^a - \Lambda \quad . \tag{3.b}$$

The action $S_{\rm H}$, contrary to $S_{\rm S}$, presents an additional non-linear local symmetry

$$\begin{aligned} \delta_\kappa \pi_2 &= \epsilon_{ab}\pi^a\kappa^b \quad , \\ \delta_\kappa \pi_a &= E\epsilon_{ab}\kappa^b \quad , \\ \delta_\kappa e^a{}_\mu &= -D_\mu\kappa^a - \frac{1}{\beta}\pi^a\epsilon_{bc}\kappa^b e^c{}_\mu \quad , \\ \delta_\kappa \omega_\mu &= \frac{\pi_2}{2\gamma}\epsilon_{ab}\kappa^a e^b{}_\mu \quad , \end{aligned} \tag{4}$$

where κ^a is the infinitesimal local parameter of the symmetry and $D_\mu\kappa^a = \partial_\mu\kappa^a + \varepsilon^a{}_b\kappa^b\omega_\mu$. The symmetry (4) $on-shell$ is related to the diffeomorphisms [5]. From Eqs. (4) one recognizes that those terms of the transformation δ_κ that do not depend on (E,β,γ) are just (linear) local κ^a-translations [2]. Thus, Eqs. (4) are the sum of pure translations plus non-linear transformations that we shall denote by $\bar\delta_\kappa$. The invariance of $S_{\rm H}$ under (4) is ensured by the fact that $\delta_\kappa \mathcal{L}_{\rm H} = \epsilon^{\mu\nu}\epsilon_{ab}\partial_\mu[(E+2\Lambda)\kappa^a e^b{}_\nu]$. This symmetry appears in (2) only at the Hamiltonian level, and in this context it has been extensively discussed in Ref.[6].

Following Refs.[1,2], the action $S_{\rm H}$ can be equivalently rewritten as an $ISO(1,1)$ gauge theory

$$S_{\rm G} = \int d^2x\, \mathcal{L}_{\rm G} = \int d^2x \left[\pi_A F^A{}_{01} + \varepsilon_{ab}\mathcal{D}_0 q^a \mathcal{D}_1 q^b \tilde{E} \right] \quad , \tag{5.a}$$

$$\tilde{E} = \frac{1}{4\gamma}(\pi_2 - \varepsilon^a{}_b\pi_a q^b)^2 - \frac{1}{2\beta}\pi_a\pi^a - \Lambda \equiv \frac{1}{4\gamma}(\pi\tilde{q})^2 - \frac{1}{2\beta}\pi_a\pi^a - \Lambda \quad , \tag{5.b}$$

$(\pi_a,\pi_2) \equiv \pi_A$ and $(T^a{}_{\mu\nu}, R_{\mu\nu}) \equiv F^A{}_{\mu\nu}$ being the components of the Lagrange multiplier and of the field strength along the Poincaré generators P_a and J, respectively [Notice that in this formulation the *zweibein* is defined as in (1) and the spacetime torsion $\mathcal{T}^a{}_{\mu\nu}$ is given in terms of the field strength components by $\mathcal{T}^a{}_{\mu\nu} = T^a{}_{\mu\nu} + \epsilon^a{}_b q^b R_{\mu\nu}$. For $q^a = 0$, $\mathcal{L}_{\rm G} \equiv \mathcal{L}_{\rm H}$ and $\mathcal{T}^a{}_{\mu\nu} \equiv T^a{}_{\mu\nu}$.]

Besides the diffeomorphism and local Poincaré invariances, the action is also invariant under the following local non-linear symmetry

$$\begin{aligned} \bar\delta_\kappa \pi_2 &= \tilde{E} q_a \kappa^a \quad , \\ \bar\delta_\kappa \pi_a &= -\tilde{E}\epsilon_{ab}\kappa^b \quad , \\ \bar\delta_\kappa e^a{}_\mu &= \epsilon_{cd}\kappa^c\mathcal{D}_\mu q^d \left(\frac{(\pi\tilde{q})}{2\gamma}\epsilon^a{}_b q^b + \frac{1}{\beta}\pi^a \right) \quad , \\ \bar\delta_\kappa \omega_\mu &= -\frac{(\pi\tilde{q})}{2\gamma}\epsilon_{ab}\kappa^a\mathcal{D}_\mu q^b \quad , \\ \bar\delta_\kappa q^a &= \kappa^a \quad , \end{aligned} \tag{6}$$

which is obviously a consequence of the non-linear symmetry (4) that the model presents in its first order formulation. In fact, up to an overall sign, Eqs. (6) with $q^a = 0$ reproduce the non-linear part of the transformations (4). Notice that, since $S_{\rm G}$ is *also* Poincaré gauge invariant, the translations and the non-linear transformations $\bar{\delta}_\kappa$ are now independent symmetries.

To simplify the generator algebra, it is convenient to consider instead of (5) the equivalent action

$$S = \int d^2x\, \mathcal{L} = \int d^2x\, [p_a \dot{q}^a + \pi_2 \dot{\omega}_1 + \pi_a \dot{e}^a{}_1 + \lambda^a J_a + \omega_0 G_2 + e^a{}_0 G_a] \tag{7.a}$$

$$G_2 = \partial_1 \pi_2 + \varepsilon_{ab} \pi^a e^b{}_1 + \epsilon_{ab} p^a q^b \quad , \tag{7.b}$$

$$G_a = \partial_1 \pi_a + \varepsilon_{ab} \pi^b \omega_1 + p_a \quad , \tag{7.c}$$

$$J_a = p_a - \tilde{E} \varepsilon_{ab} \mathcal{D}_1 q^b \quad , \tag{7.d}$$

where λ^a is a Lagrange multiplier transforming as a Lorentz vector under $ISO(1,1)$ gauge transformations. Eliminating p^a by means of the equation of motion, S becomes $S_{\rm G}$.

From (7) one sees that $\pi_A = (\pi_a, \pi_2)$ and p_a are the momenta canonically conjugate to $A_1^A = (e^a{}_1, \omega_1)$ and q^a, respectively. The remaining degrees of freedom $A_0^A = (e^a{}_0, \omega_0)$ and λ_a do not have dynamics: they play the role of Lagrange multipliers of the "Gauss' laws" $G_A = (G_a, G_2) \simeq 0$ and $J_a \simeq 0$, and the definition of their conjugate momenta provides 5 primary constraints [$\pi^{(0)}{}_A \simeq 0$ and $\pi^{(\lambda)}{}_a \simeq 0$, respectively]. As a consequence, the canonical Hamiltonian H will depend explicitly on the undetermined velocities $\dot{A}^A{}_0$ and $\dot{\lambda}^a$:

$$H = \int dx\, \mathcal{H} = \int dx\, [\pi^{(0)}{}_a \dot{e}^a{}_0 + \pi^{(0)}{}_2 \dot{\omega}_0 + \pi^{(\lambda)}{}_a \dot{\lambda}^a - \lambda^a J_a - \omega_0 G_2 - e^a{}_0 G_a] \quad . \tag{8}$$

The "Gauss' laws" related to the gauge symmetry are those associated to the Lagrange multipliers $A^A{}_0$, *i.e.* the G_A. In fact, the G_A are the $ISO(1,1)$ generators satisfying the Poincaré algebra

$$\{G_a(x), G_b(y)\} = 0 \quad , \qquad \{G_a(x), G_2(y)\} = \varepsilon_{ab} G^b \delta(x-y) \quad . \tag{9}$$

The remaining algebra involving the J_a constraints is given by

$$\{G_a(x), J_b(y)\} = 0 \quad , \tag{10.a}$$

$$\{J_a(x), G_2(y)\} = \varepsilon_a{}^b J_b \delta(x-y) \quad , \tag{10.b}$$

$$\{J_a(x), J_b(y)\} = \varepsilon_{ab} \left[\frac{1}{2\gamma} (\pi\tilde{q})(G\tilde{q}) - \frac{1}{\beta} \pi^c (G_c - J_c) \right] \delta(x-y) \quad , \tag{10.c}$$

where $(G\tilde{q}) = G_2 - \varepsilon^a{}_b G_a q^b$. As is apparent from Eqs. (10), the first class algebra of the J_a generators is non linear, and it contains in the r.h.s. structure functions, rather than structure constants. Eq. (10.c) led Strobl to conclude that our Poincaré gauge theoretical formulation of the model is redundant since "the constraint algebra is not just a representation of the Lie algebra of the gauge group". This fact, however, should not surprise as we started from an action that presents, besides the gauge symmetry, an additional non-linear local symmetry. In fact, from Eq. (10.c) it can be easily proved that the constraints J_a are precisely the generators of the symmetry (6). Consequently, they do not generate diffeomorphisms, as alleged in Refs.[3,5]. The transformations (6) are related to the diffeomorphisms *on − shell*, but are by no means the same thing. In fact, as we shall show, with the Poincaré gauge theoretic formulation one can break the former symmetry preserving the latter.

The additional non-linear symmetry can be fixed, so as to eliminate the constraints $J_a = 0$, also *without* breaking the Poincaré gauge invariance.

This opportunity is provided by the decoupling of the translations and of the non-linear symmetry $\bar{\delta}_\kappa$ that our formalism entails. One can in fact choose the *gauge-covariant* auxiliary conditions

$$\sigma_a = c\lambda_a - \pi_a \simeq 0 \quad , \tag{11}$$

where c is any arbitrary constant with dimensions of $length^{-1}$. Notice that the constraints (11) can be imposed due to the introduction of the q^a variables, namely due to the realization of the model as a gauge theory (otherwise one would have not be forced to introduce the λ_a variable). The constraints (11) make second class the J_a constraints, so that canonical (Dirac) brackets compatible with the constraints $\phi_\alpha = (\sigma_a, J_a)$ strongly equal to zero can be consistently defined [7]. For any pair $\mathcal{A}(x), \mathcal{B}(y)$ of functionals of canonical variables the Dirac brackets read

$$\begin{aligned}\{\mathcal{A}(x), \mathcal{B}(y)\}_\mathcal{D} = \{\mathcal{A}(x), \mathcal{B}(y)\} - \int du\, \{\mathcal{A}(x), \sigma_a(u)\} \frac{\varepsilon^{ab}}{\tilde{E}(u)} \{J_b(u), \mathcal{B}(y)\} \\ - \int du\, \{\mathcal{A}(x), J_a(u)\} \frac{\varepsilon^{ab}}{\tilde{E}(u)} \{\sigma_b(u), \mathcal{B}(y)\} \quad .\end{aligned} \tag{12}$$

The constraint algebra in terms of Dirac brackets then becomes

$$\{G_a(x), G_b(y)\}_\mathcal{D} = 0 \quad , \qquad \{G_a(x), G_2(y)\}_\mathcal{D} = \varepsilon_{ab} G^b \delta(x-y) \quad , \tag{13}$$

namely one is left *only* with the Poincaré algebra. This is a consequence of the fact that the constraints $\sigma_a = 0$ do not violate the gauge symmetry. Moreover the $\sigma^a = 0$, even if break the J-symmetry, do not entail the choice of a coordinate system, so that they do not break the diffeomorphisms. As a consequence the equations of motion generated by the Dirac brackets (12) with the Hamiltonian H turn out to be *identical* to the equations of motion obtained from the action given in Eqs. (5), as can be verified.

In the formulation (3) of the model, to get rid of the undesirable constraint algebra, one has to break the symmetry (4). In this case, however, one looses general covariance and alters the equations of motion.

Obviously, by choosing the "physical gauge" $q^a = 0$ instead of the condition (11), one fixes the translational part of the Poincaré symmetry, and therefore makes second class the constraints G_a (or the J_a). The remaining generators J_a (or G_a) and G_2 and the algebra (13) then reproduce the generators and the algebra of the non-linear part of the symmetry (4) plus Lorentz (G_2) transformations, and one returns to the original model, Eq. (3). In this case, however, one is left with an undesirable constraint algebra and the gauge symmetry is lost.

Thanks to our method for writing the model as a gauge theory, one can eliminate the $J_a = 0$ constraints maintaining general covariance, and one arrives at a constraint algebra that *is* a representation of the Lie algebra of the gauge group and that consequently *does* simplify the Hamiltonian structure of the model.

The canonical analysis of the Poincaré gauge theory for Liouville gravity [8] can be performed exactly in the same way (see Ref. [9], where a more exhaustive discussion of both the models is presented).

Let us now analyze the other doubts raised by Strobl. The second main criticism of Ref.[3] is that our approach – obtained by introducing the Poincaré coordinates q^a and the Poincaré gauge connection A_μ – is "trivially equivalent", at the classical level, to the one obtained by considering the original action in a first order formalism, with *vielbein* $V^a{}_\mu$ and spin connection as independent variables, and he writes

$$L(q, e_\mu, \omega_\mu) \sim L(V_\mu, \omega_\mu) \quad . \tag{14}$$

We certainly agree on the equivalence, it was our intention to provide an *equivalent* formulation. However, the theory defined in the l.h.s. of Eq. (14) is a gauge theory, the one in the r.h.s. is not. In particular, in the standard first order formalism, the *vielbein* cannot be written in terms of Poincaré gauge potentials A_μ for which the transformation law is the standard $\delta A_\mu = -\partial_\mu u - [A_\mu, u]$ [With the exceptions of pure gravity in 3 dimensions [10] and of pure 2-dimensional "black-hole" gravity when treated as a gauge theory of the *extended* Poincaré group [11]]. To appreciate this point, it is instructive to draw an analogy with classical electrodynamics. To formulate electrodynamics as a gauge theory, unphysical extra-degrees of freedom are needed, the longitudinal components of the gauge potentials. In some "physical gauge" such unphysical components can be gauged away. In addition, it is well known that with a suitable shift of gauge potentials, the Maxwell action is "trivially equivalent" to the action of a massless Proca field. However, in this case the gauge structure of the Maxwell action has been unavoidably lost. Moreover, in a path integral, the functional determinant of the coordinate redefinition that leads from the Maxwell to the Proca fields, in the Lorentz gauge would just give an inessential normalization factor, precisely as the coordinate redefinition proposed by Strobl for our formalism in the physical gauge. Nevertheless, for an arbitrary gauge condition $F(q, e_\mu, \omega_\mu) = 0$, such redefinition might non-trivially influence the functional Dirac-δ of the gauge condition and the Fadeev-Popov determinant, as it happens in the usual formulation of Yang–Mills theories with an arbitrary gauge choice $F(A_\mu) = 0$.

The author of Ref.[3], therefore, should not be "struck" by the redundant and unphysical degrees of freedom that our formalism entails: this is a characteristic feature of gauge theories.

Other two remarks in Strobl's comment deserve an answer. The first one concerns a technical point related to the possibility of "neutralizing" the momentum part of the Poincaré transformations in any matter multiplet Φ through a suitable redefinition of Φ. We explained exhaustively this point at the end of Sect. 3 in Ref.[2].

Strobl considers, as an example, the Poincaré gauge invariant scalar field action in 4 dimensions that we provided in Ref.[1]. He uses what we called in Ref.[2] the transformation to the zero momentum representation.

Following Ref.[2], we can be more general by considering any matter multiplet in any dimension: if Φ is a matter multiplet transforming according to any given representation (J,P) of the Poincaré group, then the multiplet $\tilde{\Phi} = (\mathbf{1} - q^a P_a)\Phi$ transforms according to the zero-momentum representation (J,0), namely

$$\delta\Phi^A = (\alpha \cdot J + \rho \cdot P)^A{}_B \Phi^B \Longrightarrow \delta\tilde{\Phi}^A = (\alpha \cdot J)^A{}_B \tilde{\Phi}^B \quad . \tag{15}$$

From this property Strobl concludes that the translational part of the Poincaré gauge group is superfluous in the gauge theory, and that the theory is equivalent to the one obtained by considering the matter multiplet in the zero-momentum representation and the *vielbein* $V^a{}_\mu$ as an independent variable. The theory one gets is certainly equivalent at the classical level, but it is no longer a gauge theory, because in this case the *vielbein* is not expressed in terms of Poincaré gauge potentials.

The last of Strobl's comments concerns an incorrect statement in the Appendix of Ref.[2]. He has an old version of our paper. In fact, in the first release of the preprint the Appendix was mistaken, but it was promptly replaced by a correct one about a week after the submission: at least since October 1992, in the version of our preprint that can be found in all the electronic libraries, the Appendix is correct.

Finally, we take the opportunity to make a remark on our papers. In Refs.[1,2,9,12] the whole analysis was performed at the classical level. Concerning possible developments at the quantum level, we always used the conditional form. Certainly we do not expect that our procedure solves the huge problems that arise in quantum gravity. The real problems of a quantum theory of gravity presumably cannot be overcome just by a gauge theoretical description of the theory. Nevertheless, such a description could provide an alternative to the standard approach that could be worth to investigate.

REFERENCES

[1] G. Grignani and G. Nardelli, *Phys. Rev.* **D45** (1992) 2719.

[2] G. Grignani and G. Nardelli, "Poincaré Gauge Theories for Lineal Gravities", Preprint DFUPG-57-1992/UTF-266-1992.

[3] T. Strobl, "Comment on Gravity and the Poincaré Group", Preprint TUW-01-1993.

[4] M. O. Katanaev and I. V. Volovich, *Ann. Phys.* (N.Y.) **197** (1986) 1.

[5] T. Strobl, *Int. J. Mod. Phys.* **8**, (1993) 1883.

[6] H. Grosse, W. Kummer, P. Presnaider and D. J. Schwarz, *J. Math. Phys.* **33** (1992) 3892.

[7] See for example A. Hanson, T. Regge and C. Teitelboim, "Constrained Hamiltonian Systems", Accademia Nazionale dei Lincei, Roma 1979.

[8] C. Teitelboim, *Phys. Lett.* **B126**, 41 (1983), and in *Quantum Theory of Gravity*, S Christensen ed. (Adam Higler, Bristol, 1984); R. Jackiw in *Quantum Theory of Gravity*, S Christensen ed. (Adam Higler, Bristol, 1984) and *Nucl. Phys.* **B252**, 343 (1985). For a first order formulation of Liouville gravity as a de–Sitter gauge theory see for instance K. Isler and C. A. Trugenberger, *Phys. Rev. Lett.* **63**, 834 (1989).

[9] G. Grignani and G. Nardelli, "Canonical Analysis of Poincaré Gauge Theories for Two Dimensional Gravity", Preprint DFUPG-74-1993/UTF-292-1993, March 1993.

[10] A. Achucarro and P. Townsend, *Phys. Lett.* **B180** (1986) 85; E. Witten, *Nucl. Phys.* **B311** (1988) 46. To this purpose it should be noticed that even in 3 dimensions, if matter

interactions are present, the component of the gauge potential along the momentum generators cannot be interpreted as *dreibein* without encountering inconsistencies.

[11] D. Cangemi and R. Jackiw, "Poincaré Gauge Theory for Gravitational Forces in 1+1 Dimensions", MIT preprint CTP # 2165 (*Ann. of Phys.*, in press) and references therein.

[12] G. Grignani and G. Nardelli, *Phys. Lett.* **B264** (1991) 45; ibid, *Nucl. Phys.* **B370** (1992) 491; ibid, *Phys. Lett.* **B300** (1993) 38.